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Transient wave propagation in bubbly liquids

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A theoretical and numerical investigation of the propagation of one-dimensional waves in a bubbly liquid is presented. A variational formulation of the problem is used that yields both the linear-momentum equation and the equation that describes the oscillations of the bubbles. The compressibility of the liquid is taken into account in the formulation. The thermal dissipation is treated by solving the energy-balance equations simultaneously with the mechanical equations. Solutions are obtained by a finite-difference procedure and are compared to the experimental data of Kuznetsov *et al.* and Noordzij & van Wijngaarden. In some cases quite good agreement is obtained, but in others substantial errors are found. It is suggested that the observed discrepancies may be due to the breakup of the bubbles in the case of very large amplitude disturbances; the fact that the formulation does not include relative motion between the liquid and the bubbles; and possible non-planarity effects in the experiments.

1. Introduction

It is well known that the presence of bubbles of gas in a liquid greatly alters the character of the propagation of pressure waves through the liquid. This fact has important implications for sonar applications (Urick 1975), for many problems associated with nuclear-reactor technology (Walchli & West 1964), and for certain geophysical problems (Kieffer 1977).

In the case of steady harmonic waves of small amplitude, it has been shown both experimentally and analytically that the phase velocity of compressive waves drops and the attenuation rises as the frequency of the wave increases. These effects become very pronounced as the frequency of the wave approaches the natural frequency associated with expansional oscillations of the bubbles (Carstensen & Foldy 1947; Fox, Curley & Larson 1955; Macpherson 1957; Silberman 1957; McWilliam & Duggins 1969; van Wijngaarden 1972*a*, *b*; Medwin 1974; Plesset & Prosperetti 1977; Drumheller & Bedford 1979*a*).

The dispersion and dissipation that are observed in small-amplitude harmonic waves will, of course, affect the propagation of transient waves in bubbly liquids. Furthermore, when finite-amplitude waves are considered, non-linear effects occur which have been shown to have a crucial influence on the qualitative as well as the quantitative characteristics of the waves (Campbell & Pitcher 1958; Crespo 1969; Eddington 1970; Noordzij 1973; van Wijngaarden 1972*a*, *b*, Gel'fand *et al.* 1974). In the most extensive investigation that has been carried out thus far, Kuznetsov

et al. (1978) have shown that, depending upon the strength of the input pulse and the dissipative character of the medium, a transient pulse may propagate in a bubbly liquid in the form of a single soliton, multiple solitons, an oscillatory wave packet, an oscillatory shock wave, or a non-oscillatory shock wave.

This great diversity of behaviour of transient pulses in bubbly liquids was predicted theoretically before it was actually observed. By writing the governing equations in terms of a perturbation of the pressure, and retaining terms through the second order, the equations have been shown to reduce to the Korteweg–de Vries equation (Benjamin 1966). When dissipation is included, the second-order equations reduce to the Burgers–Korteweg–de Vries equation (van Wijngaarden 1972*a*, Nakoryakov, Sobolev & Shreiber 1972, Noordzij 1973). Kuznetsov *et al.* (1978) showed that, for appropriate values of the nonlinearity and dissipation parameters, the Burgers–Korteweg–de Vries equation correctly predicts all of the qualitative behaviours that were observed in their experiments. However, they did not show comparisons of their measured pressure distributions with theoretical predictions.

In this paper, a theoretical and numerical investigation of the propagation of transient pulses in bubbly liquids is presented that differs from previous works in several ways. Previously, the governing equations have been obtained by deriving the linear-momentum equation for the mixture independently of the equation for the expansional motion of the bubbles (van Wijngaarden 1972*a*). In this paper, a variational formulation is used that yields the coupled linear-momentum and expansional equations directly. The variational formulation has been applied to small-amplitude waves by Drumheller & Bedford (1979*a*, 1980*a*), and it was shown that the linearized equations reduce to those that have been used previously.

The transfer of heat between the gas and the liquid results in an important dissipation effect in the propagation of waves in bubbly liquids (van Wijngaarden 1972*a*). In previous treatments, the heat-transfer problem was solved first to determine a damping coefficient, which was then used in the equations of motion. In this paper, the balance-of-energy equation is derived, and is solved simultaneously with the mechanical equations of motion as a coupled problem. This procedure will be shown to give improved agreement with the measured wave amplitude.

In many applications involving wave propagation in bubbly liquids, it can be safely assumed that the liquid is incompressible owing to the relatively greater compressibility of the gas. However, in some cases of shock-wave propagation, the compression wave that propagates through the liquid (followed by the much-slower disturbance associated with the compression of the bubbles) will be of interest. In this paper the compressibility of the liquid is included in the formulation, and an example is presented that illustrates this effect. (A further consideration was that the algorithm that was used to solve the governing equations required that the liquid be treated as compressible.)

Previous numerical solutions for transient pulse propagation in bubbly liquids have been based on the second-order equations. In this investigation the general nonlinear forms of the equations have been used. Solutions for the pressure distributions caused by transient pulses were determined by the method of finite differences and compared to the experimental distributions reported by Kuznetsov *et al.* (1978) and Noordzij & van Wijngaarden (1974).

2. Theory

Consider a mixture of two constituents, a liquid and a gas. The mass density of the ξth constituent ($\xi = \mathrm{f}$ for the liquid and $\xi = \mathrm{g}$ for the gas) will be denoted by $\bar{\rho}_\xi$. The volume fraction of the ξth constituent will be denoted by ϕ_ξ, where

$$\sum_\xi \phi_\xi = \phi_\mathrm{f} + \phi_\mathrm{g} = 1. \tag{1}$$

The partial density ρ_ξ, the mass of the ξth constituent per unit volume of the mixture, is related to the mass density and the volume fraction by

$$\rho_\xi = \phi_\xi \bar{\rho}_\xi. \tag{2}$$

The equation of conservation of mass for each constituent is

$$\dot{\rho}_\xi + \rho_\xi \operatorname{div} \mathbf{v} = 0, \tag{3}$$

where $\mathbf{v}$ is the velocity of the mixture, and a dot denotes the material derivative:

$$\dot{\rho}_\xi = \frac{\partial}{\partial t}\rho_\xi + \mathbf{v}.\operatorname{grad}\rho_\xi. \tag{4}$$

Alternatively, (3) can be written

$$\rho_\xi J = \rho_{\xi_0}, \tag{5}$$

where J is the Jacobian and ρ_{ξ_0} is the partial density in a reference configuration. When the equations (5) for $\xi = \mathrm{g}$ and $\xi = \mathrm{f}$ are summed, the equation of conservation of mass for the mixture

$$\rho J = \rho_0 \tag{6}$$

is obtained, where $\rho = \sum_\xi \rho_\xi$.

The variations of the velocity and the Jacobian are (Finlayson 1972)

$$\delta\mathbf{v} = \delta\dot{\mathbf{x}}, \quad \delta J = J \operatorname{div} \delta\mathbf{x}, \tag{7}$$

where $\mathbf{x}$ is the position vector of a material point of the mixture. The variation of a function $\Psi(\mathbf{x}, t)$ holding $\mathbf{x}$ fixed is

$$\delta\Psi|_\mathbf{x} = \delta\Psi - \delta\mathbf{x}.\operatorname{grad}\Psi. \tag{8}$$

Hamilton's extended principle can be written

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W)\, dt = 0, \tag{9}$$

where t_1 and t_2 are arbitrary times, T and U are the kinetic and potential energies, and δW is the virtual work. The governing equations will be derived by applying (9) to a material volume v of the mixture (Bedford & Drumheller 1978; Drumheller & Bedford 1979*a*, *b*, 1980*a*).

The kinetic energy consists of the kinetic energy of translation plus the kinetic energy of the liquid that is displaced radially as the bubbles expand or contract. The latter energy will be evaluated by assuming that the radial velocity distribution in the neighbourhood of a given bubble, relative to the bubble, can be approximated by the velocity distribution for a single bubble in an unbounded incompressible fluid, $v_r = (R/r)^2 \dot{R}$, where v_r is the radial velocity of the liquid at a distance r from the centre of the bubble, R is the bubble radius, and $\dot{R} = dR/dt$ (van Wijngaarden 1972*a*). This

assumption requires that the distance between bubbles be large compared to R. Furthermore, since the liquid is assumed to be compressible in the present analysis, this assumption also requires that changes in the liquid density be small over distances that are of the order of the distance between bubbles.

Since $\frac{4}{3}\pi R^3\bar{\rho}_g$ is constant, the kinetic energy of the liquid surrounding a single bubble is approximated by

$$\int_R^\infty \tfrac{1}{2}\bar{\rho}_f v_r^2 4\pi r^2 dr = \tfrac{2}{9}\pi R^5 \frac{\bar{\rho}_f}{\bar{\rho}_g^2}\dot{\bar{\rho}}_g^2. \tag{10}$$

By multiplying (10) by the number of bubbles per unit volume $\phi_g/\frac{4}{3}\pi R^3$, to obtain the kinetic energy per unit volume associated with bubble oscillation, the total kinetic energy of the mixture contained in a material volume v can be written as

$$T = \int_v \left(\tfrac{1}{2}\rho \mathbf{v}.\mathbf{v} + \frac{1}{6}\frac{R^2\phi_g\bar{\rho}_f}{\bar{\rho}_g^2}\dot{\bar{\rho}}_g^2\right) dv. \tag{11}$$

The virtual work is written

$$\delta W = \sum_\xi \int_v -\phi_\xi P_\xi \frac{\delta\bar{\rho}_\xi}{\bar{\rho}_\xi} dv + \int_v \phi_g Q \frac{\delta\bar{\rho}_g}{\bar{\rho}_g} dv + \sum_\xi \int_v \rho_\xi \mathbf{f}_\xi.\delta\mathbf{x}\, dv + \int_s \mathbf{T}.\delta\mathbf{x}\, ds. \tag{12}$$

The term P_ξ is the pressure of the ξth constituent, $\phi_g Q\delta\bar{\rho}_g/\bar{\rho}_g$ is a virtual-work term which will be used to introduce the bubble damping, $\mathbf{f}_\xi$ is the external body force density acting on the ξth constituent, and $\mathbf{T}$ is a specified traction vector that is assumed to act upon a portion s' of the surface s of v. (It is ssumed that the displacement of the mixture is specified on the remainder of s, so that $\delta\mathbf{x} = 0$ except on s'.) Work expressions for all of the forces acting on the mixture are given in (12). Consequently $\delta U = 0$.

Equations (1) and (5) will be introduced into the variational principle (9) as constraints by using the method of Lagrange multipliers. The multipliers are denoted by λ and μ_ξ, respectively. Thus, Hamilton's extended principle for the mixture is written

$$\int_{t_1}^{t_2}\Bigg[\delta\int_v\left(\tfrac{1}{2}\rho\mathbf{v}.\mathbf{v} + \tfrac{1}{6}\rho_g\frac{R^2\bar{\rho}_f}{\bar{\rho}_g^3}\dot{\bar{\rho}}_g^2\right)dv + \sum_\xi\int_v\left(-\phi_\xi P_\xi\frac{\delta\bar{\rho}_\xi}{\bar{\rho}_\xi} + \rho_\xi\mathbf{f}_\xi.\delta\mathbf{x}\right)dv + \int_v\phi_g Q\frac{\delta\bar{\rho}_g}{\bar{\rho}_g}dv$$
$$-\int_v \lambda\delta\left(\sum_\xi\phi_\xi - 1\right)\Big|_x dv + \sum_\xi\int_v\mu_\xi\delta\left(J - \frac{\rho_{\xi_0}}{\rho_\xi}\right)dv + \int_{s'}\mathbf{T}.\delta\mathbf{x}\, ds\Bigg]dt = 0. \tag{13}$$

The resulting equations of motion obtained from (13), corresponding to the variations $\delta\mathbf{x}$, $\delta\bar{\rho}_g$, $\delta\bar{\rho}_f$, and $\delta\phi_\xi$, are†

$$\rho\dot{\mathbf{v}} = \rho\mathbf{f} - \operatorname{grad}\left(\sum_\xi \mu_\xi J\right), \tag{14}$$

$$\bar{\rho}_g^2\left(\overline{\frac{R^2\dot{\bar{\rho}}_f}{3\bar{\rho}_g^3}\dot{\bar{\rho}}_g}\right) + \frac{11}{18}\frac{R^2\bar{\rho}_f}{\bar{\rho}_g^2}\dot{\bar{\rho}}_g^2 = \frac{\mu_g J}{\phi_g} - P_g + Q \tag{15}$$

$$\frac{1}{6}\frac{\phi_g R^2\dot{\bar{\rho}}_f}{\phi_f\bar{\rho}_g^2}\bar{\rho}_g^2 + \frac{\mu_f J}{\phi_f} - P_f = 0, \tag{16}$$

$$\frac{\mu_\xi J}{\phi_\xi} = \lambda, \tag{17}$$

† In evaluating the variations in (13) it must be noted that R depends upon $\bar{\rho}_g$. It must also be noted that since v is a material volume $\delta\int_v\rho_\xi\Psi\,dv = \int_v\rho_\xi\delta\Psi\,dv$ (see Drumheller & Bedford 1980*b*). The variations are assumed to vanish at $t = t_1$ and $t = t_2$.

and the boundary condition on s' is

$$\mathbf{T} = -\sum_{\xi} \mu_{\xi} J \mathbf{n}, \tag{18}$$

where $\mathbf{n}$ is an outward unit vector normal to s, and $\rho \mathbf{f} = \sum_{\xi} \rho_{\xi} \mathbf{f}_{\xi}$.

By using (17) to eliminate $\mu_{\xi} J$ from (14)–(16) and (18), they can be written

$$\rho \dot{\mathbf{v}} = \rho \mathbf{f} - \operatorname{grad} \lambda, \tag{19}$$

$$\frac{\ddot{\bar{\rho}}_{\mathrm{g}}}{\bar{\rho}_{\mathrm{g}}} + \left(\frac{\phi_{\mathrm{g}}}{\phi_{\mathrm{f}}} - \frac{11}{3}\right) \frac{\dot{\bar{\rho}}_{\mathrm{g}}^2}{2\bar{\rho}_{\mathrm{g}}^2} + \frac{\dot{\bar{\rho}}_{\mathrm{f}} \dot{\bar{\rho}}_{\mathrm{g}}}{\bar{\rho}_{\mathrm{f}} \bar{\rho}_{\mathrm{g}}} = \frac{3}{R^2 \bar{\rho}_{\mathrm{f}}} (P_{\mathrm{f}} - P_{\mathrm{g}} + Q), \tag{20}$$

$$\lambda = P_{\mathrm{f}} - \frac{\phi_{\mathrm{g}} R^2 \bar{\rho}_{\mathrm{f}}}{6 \phi_{\mathrm{f}} \bar{\rho}_{\mathrm{g}}^2} \dot{\bar{\rho}}_{\mathrm{g}}^2, \tag{21}$$

$$\mathbf{T} = -\lambda \mathbf{n} \quad \text{on} \quad s'. \tag{22}$$

Equation (19) is the linear-momentum equation, (20) is the expansional equation, and they are coupled through (21). When (20) is written in terms of R, it becomes

$$R\ddot{R} + \frac{3}{2}\left(1 - \frac{\phi_{\mathrm{g}}}{\phi_{\mathrm{f}}}\right) \dot{R}^2 + \frac{\dot{\bar{\rho}}_{\mathrm{f}}}{\bar{\rho}_{\mathrm{f}}} R\dot{R} + \frac{Q}{\bar{\rho}_{\mathrm{f}}} = \frac{P_{\mathrm{g}} - P_{\mathrm{f}}}{\bar{\rho}_{\mathrm{f}}}. \tag{23}$$

Note that when the liquid is incompressible ($\bar{\rho}_{\mathrm{f}} = \text{const.}$), the bubble damping term Q is assumed to be zero, and the bubble concentration is negligible ($\phi_{\mathrm{g}}/\phi_{\mathrm{f}} \to 0$), (23) reduces to the Rayleigh–Plesset equation (Plesset & Prosperetti 1977). Equation (23) is also similar in form to the relation proposed by Nigmatulin (1979) in which empirical corrections for bubble concentration are included.

For the particular examples considered in this work, the $\phi_{\mathrm{g}}/\phi_{\mathrm{f}}$ correction to the second term in (23) is negligible; however, the effects of $\dot{\bar{\rho}}_{\mathrm{f}}/\bar{\rho}_{\mathrm{f}}$ in the third term in (23) are significant. This term results in the propagation of pressure waves in the liquid component. As will be shown, these pressure waves result in negligible density variations over a distance equal to the bubble spacing. Consequently, the assumptions leading to (10) are not violated.

Dissipation mechanisms that are present in dynamic processes in bubbly liquids include viscous dissipation in the liquid, irreversible transfer of heat from the gas to the liquid, also – particularly at frequencies above the resonance frequency of the bubbles – the radiation of compressional waves in the liquid owing to the oscillations of the bubbles (Poritsky 1952; Devin 1959; van Wijngaarden 1972*a*), and evaporation and condensation of any liquid vapour that is in solution with the gas constituent that forms the bubbles.

The last effect has been ignored by previous authors. As in the present case, these authors studied systems in which the liquid component was water. Neglecting this effect seems justified since accurate predictions of both sound speed and attenuation of time-harmonic pressure waves can be achieved without inclusion of this effect (Drumheller & Bedford 1979*a*). The remaining mechanisms have previously been included in studies of wave propagation in bubbly liquids by evaluating an effective damping coefficient Q in (23) (van Wijngaarden 1972*a*; Noordzij & Wijngaarden 1974). In this paper, the viscous-dissipation and radiation mechanisms have been treated in this way. However, in order to obtain agreement with the experimental results, it was found to be necessary to treat the heat-transfer mechanism in a more elaborate way.

Consider the compression of a bubbly liquid. The compression causes the temperature of the gas in the bubbles to rise above the temperature of the surrounding liquid. As a result, heat flows from the bubbles to the liquid. Since this process is not completely reversible, some of the mechanical energy required to compress the bubbly liquid is not recoverable. This is the mechanism of the dissipation.

Since the thermal damping is associated with the temperature of the gas and liquid and the flow of heat between them, energy equations will be introduced into the theory that will allow for different temperatures and energy exchange between the liquid and the gas (Crespo 1969; Drumheller & Bedford 1980*a*).

The energy equations will be obtained by the method described by Drumheller & Bedford (1979*b*). By noting that the definition of virtual work introduced in (12) must also define the mechanical-work expressions that should appear in the energy equations, the energy-balance relationships can be written

$$\rho_{\mathrm{g}}\dot{E}_{\mathrm{g}} = \phi_{\mathrm{g}}(P_{\mathrm{g}}-Q)\frac{\dot{\bar{\rho}}_{\mathrm{g}}}{\bar{\rho}_{\mathrm{g}}}+\rho_{\mathrm{g}}r_{\mathrm{g}}+u \tag{24}$$

$$\rho_{\mathrm{f}}\dot{E}_{\mathrm{f}} = \phi_{\mathrm{f}}P_{\mathrm{f}}\frac{\dot{\bar{\rho}}_{\mathrm{f}}}{\bar{\rho}_{\mathrm{f}}}-\operatorname{div}\mathbf{q}_{\mathrm{f}}+\rho_{\mathrm{f}}r_{\mathrm{f}}-u, \tag{25}$$

where E_ξ is the specific internal energy of the ξth constituent, r_ξ is the specific external heat supply (by radiation, for example), u is the energy exchange between constituents, and $\mathbf{q}_{\mathrm{f}}$ is the heat flux within the liquid. The heat flux $\mathbf{q}_{\mathrm{f}}$ should not be confused with the heat flux between the gas and the liquid, which is represented by u. (The heat flux $\mathbf{q}_{\mathrm{g}}$ within the gas will not be included because the bubbles do not touch and thus do not provide a path for direct heat transfer between bubbles.)

3. Constitutive assumptions

The heat flux $\mathbf{q}_{\mathrm{f}}$ and the external heat supplies r_ξ will be neglected. The gas in the bubbles is assumed to be an ideal gas so that

$$P_{\mathrm{g}} = R_u\bar{\rho}_{\mathrm{g}}T_{\mathrm{g}}, \tag{26}$$

where T_{g} is the absolute gas temperature and R_u is the gas constant. The specific internal energy is assumed in the form

$$E_{\mathrm{g}} = C_{v\mathrm{g}}T_{\mathrm{g}}, \tag{27}$$

where $C_{v\mathrm{g}}$ is a constant denoting the specific heat at constant volume of the gas. Thus the energy equation for the gas, (24), becomes

$$\rho_{\mathrm{g}}C_{v\mathrm{g}}\dot{T}_{\mathrm{g}} = \phi_{\mathrm{g}}(P_{\mathrm{g}}-Q)\frac{\dot{\bar{\rho}}_{\mathrm{g}}}{\bar{\rho}_{\mathrm{g}}}+u. \tag{28}$$

The liquid constituent will be assumed to obey the pressure-density relation

$$P_{\mathrm{f}} = K\frac{\bar{\rho}_{\mathrm{f}}-\bar{\rho}_{\mathrm{fr}}}{\bar{\rho}_{\mathrm{f}}}, \tag{29}$$

where K is the liquid bulk stiffness and $\bar{\rho}_{\mathrm{fr}}$ is a reference density. The time derivative of the specific internal energy is assumed to be of the form

$$\dot{E}_{\mathrm{f}} = P_{\mathrm{f}}\frac{\dot{\bar{\rho}}_{\mathrm{f}}}{\bar{\rho}_{\mathrm{f}}^2}+C_{v\mathrm{f}}\dot{T}_{\mathrm{f}}, \tag{30}$$

where C_{vf} is a constant denoting the specific heat at constant volume of the liquid and T_f is the liquid temperature. When (30) is substituted into the energy equation for the liquid, (25), one obtains

$$\rho_f C_{vf} \dot{T}_f = -u. \tag{31}$$

To complete the material descriptions, constitutive expressions for Q and u are required. The damping term Q has been evaluated in the present work by using the acoustic approximation described by Noordzij & van Wijngaarden (1974). The thermal damping part of the term was of course not included.

An expression for u was obtained by considering the local heat conduction in the neighbourhood of a single bubble. The expression was derived from the solution for the temperature field in and around a bubble whose pressure is oscillating with a harmonic frequency ω.† After removal of the harmonic time dependence, the solution of the linear heat-conduction equation gives the distribution of local temperature θ inside the bubble as

$$\theta - T_g = \frac{R}{r}(\theta_w - T_g)\frac{\sinh(i^{\frac{1}{2}}\alpha_g r)}{\sinh(i^{\frac{1}{2}}\alpha_g R)}, \tag{32}$$

where θ_w is the temperature at the bubble surface (Drumheller & Bedford 1979*a*, equation (56)). The temperature distribution outside the bubble is

$$\theta - T_f = \frac{R}{r}(\theta_w - T_f)\exp[i^{\frac{1}{2}}\alpha_f(R-r)], \tag{33}$$

(Drumheller & Bedford 1979*a*, equation (61)). The parameters α_ξ are given by

$$\alpha_\xi^2 = \frac{\omega\bar{\rho}_\xi C_{p\xi}}{k_\xi}, \tag{34}$$

where $C_{p\xi}$ and k_ξ are the specific heat at constant pressure and the thermal conductivity of the ξth constituent, and $i = \sqrt{-1}$.

At the bubble surface, the temperature and heat flux must be continuous. By applying these two conditions to (32) and (33), the following expression for θ_w is obtained:

$$\theta_w = \frac{k_g \alpha T_g - k_f \beta T_f}{k_g \alpha - k_f \beta}, \tag{35}$$

where

$$\alpha = i^{\frac{1}{2}}\alpha_g R \operatorname{cotanh}(i^{\frac{1}{2}}\alpha_g R) - 1, \tag{36}$$

$$\beta = i^{\frac{1}{2}}\alpha_f R + 1. \tag{37}$$

Then the heat flux across the bubble surface is

$$\frac{k_f}{R}\beta(\theta_w - T_f). \tag{38}$$

The magnitude of this quantity, when multiplied by the surface area of an individual bubble and the number of bubbles per unit volume, gives u as

$$u = -\frac{3\phi_g k_f}{R^2}|\beta|(\theta_w - T_f). \tag{39}$$

† This approach is valid only for small-amplitude oscillations since a moving boundary-value problem is not solved. This approximation may result in significant effects on the computed thermal damping of finite-amplitude waves.

This expression is complete provided an estimation of frequency ω is known. In the present analysis the bubbly liquid does not in general exhibit a harmonic response, but rather a transient response, which can be regarded as a spectrum of harmonic components. The expression for ω will be chosen to estimate the value of the dominant frequency in this spectrum.

The variable $\bar{\rho}_g$ is used to estimate a value for ω at each position and time by the relationship

$$\omega = \frac{2}{\pi}\left|\frac{\dot{\bar{\rho}}_g}{\bar{\rho}_g - \bar{\rho}_{g0}}\right|. \tag{40}$$

The parameter $\bar{\rho}_{g0}$ is the initial value of $\bar{\rho}_g$. At any given time $\dot{\bar{\rho}}_g$ is viewed as a maximum value of the dominant harmonic. Therefore $(2/\pi)\,|\dot{\bar{\rho}}_g|$ represents the magnitude averaged over one-half of an oscillation. This averaged value is used in (40). Equation (40) allows the possibility of infinite values of ω. This limit clearly corresponds to adiabatic behaviour.

As noted previously, the inclusion of the energy relations (24) and (25) is an alternative approach to modelling the thermal damping. The conventional approach is to add an acoustical approximation term to the definition of the damping term Q. This approximation is achieved by constructing a separate solution to a system of energy equations that are similar to those presented in this work. One of the major differences in comparison with the present approach is that (40) is replaced by the assumption that ω is equal to the resonance frequency of the bubble at ambient conditions.

4. Numerical analysis

Equations (1), (2), (5), (6), (19)–(21), (26), (28)–(30), (35), (37) and (39) provide a system of equations in the variables ϕ_ξ, λ, ρ_ξ, ρ, $\mathbf{v}$, $\bar{\rho}_\xi$, P_ξ, T_ξ, u, θ_w, and ω. These equations have been solved by using the one-dimensional explicit Lagrangian finite-difference wave-propagation code WONDY IV; the differencing procedure that is used is described by Lawrence & Mason (1971).

Briefly, the programme is designed to solve (6) and (19) together with an equation of state for λ. In solving the bubbly liquid problem, the remaining equations have been treated as equations of state. Thus the programme used (6) and (19) to update ρ and $\mathbf{v}$ from a given time to the next time step. The equations of state were then solved simultaneously to determine ϕ_ξ, λ, ρ_ξ, $\bar{\rho}_\xi$, P_ξ, T_ξ, u, θ_w and ω at the next time step. In doing so, (20), (28) and (30) were integrated numerically by using an auxiliary ordinary differential-equation solver STEP (Shampine & Gordon 1974).

Normally, the finite-difference equations contained in the programme WONDY IV also include an artificial viscosity. This term is designed to smooth out wave profiles and eliminate the possibility of shock-wave discontinuities. In the present investigation this artificial smoothing has not been required. It is replaced by the physical smoothing process represented by the bubble response (20). Thus the results discussed in § 5 were obtained from calculations without artificial viscosity. These results were also unaltered by refinements in the mesh spacings and time step.

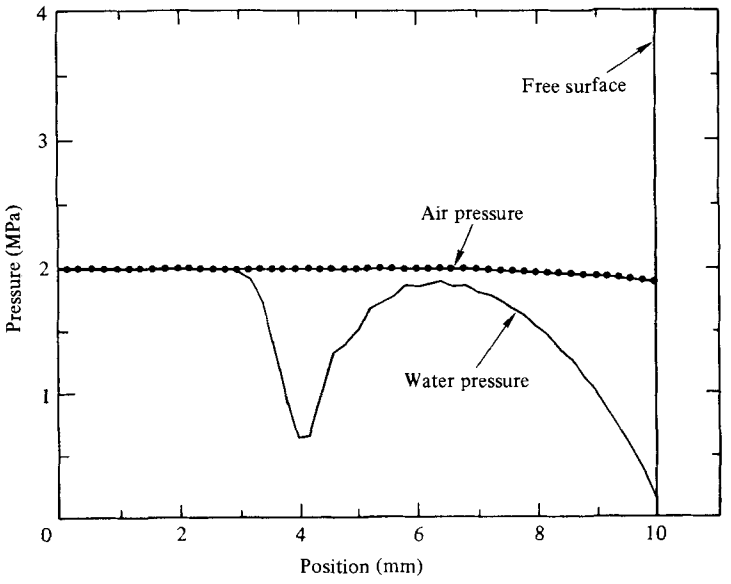

FIGURE 1. Pressure profiles in bubbly water during expansion; time = 4·1 μs.

5. Results

A solution will first be presented that illustrates the effects of the compressibility of the liquid on the early-time response of a bubbly liquid to a transient excitation. The problem models a mixture of air bubbles and water that occupies the volume between two vertical infinite parallel surfaces that are 10 mm apart. The bubbles are of 1 mm radius and initially occupy 10 % of the mixture volume. The initial pressure is set equal to 2·0 MPa. At time zero the right-hand surface is assumed to move to the right such that the mixture is subjected to a boundary pressure of 0·1 MPa. The opposite surface is held fixed.

If the liquid were considered to be incompressible, a single pressure-release wave, propagating at the sound velocity based on the static compressibility of the mixture U_{H}, would travel away from the right-hand wall. For this example, the value of U_{H} is 180 m/s. Based on this velocity, the mixture at the fixed wall would not respond to the pressure drop until 55 μs of time had elapsed.

In contrast to this behaviour, the response including the compressibility of the liquid that is predicted by the present theory is illustrated in figures 1 and 2. The pressure profiles for both P_{f} and P_{g} are shown at two different times.† In figure 1 the pressure-release wave has propagated to the position 3·7 mm at a rate of 1·5 km/s, the sound speed in the liquid. In response to the pressure drop in the liquid, the bubbles have expanded slightly, resulting in a small pressure drop within the bubbles. This slight expansion of the bubbles in turn repressurizes the stiff liquid to form a broad dome behind the pressure-release wave. At the later time shown in figure 2, this pressure dome has grown larger, and owing to the large inertial effects associated with

† It must be noted that the assumption that variations in the density of the liquid must be small over distances of the order of the distance between bubbles is violated in this example. This fact could cause error in the distribution of P_{f}, but the qualitative features of the example should not be altered.

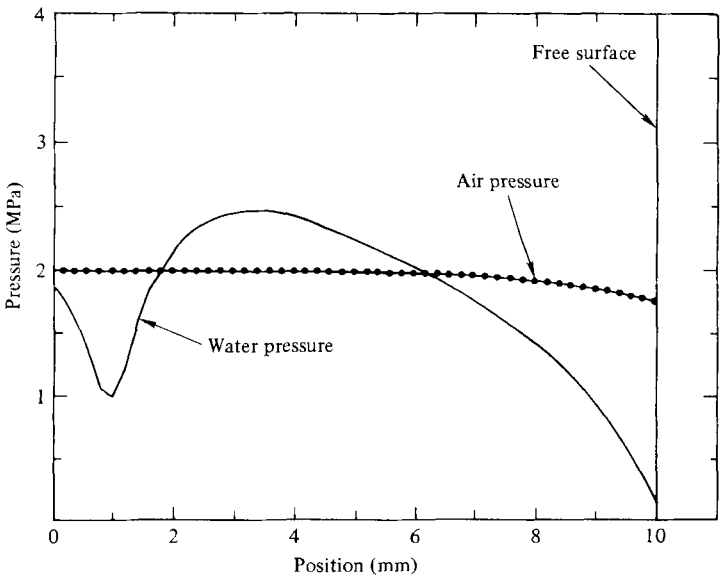

FIGURE 2. Pressure profiles in bubbly water during expansion; time = 6·05 μs.

the expansion of the bubbles, the pressure in the liquid behind the release wave has increased to a value greater than the original pressure. At a slightly later time (7 μs) the release wave reaches the fixed wall. Thus in this case the total volume of the mixture responds to the original pressure drop after 7 μs instead of after 55 μs as in the incompressible liquid assumption.

While the previous calculations demonstrate significant differences due to the inclusion of liquid compressibility, there are no early-time experimental data with which comparisons can be made. In the case of the late-time response, comparisons can be made with the data of Kuznetsov *et al.* (1978) and Noordzij & van Wijngaarden (1974), which were based upon measurements of the pressure profile of a compression wave propagating down a tube which contained a bubbly water–glycerine solution.

The results of Kuznetsov *et al.* (1978) will be considered first.† The pressure disturbance was generated by a pressure chamber mounted above the bubbly solution. Two series of tests were conducted, one using carbon dioxide bubbles and one using helium bubbles, in order to obtain a large range of nonlinear and dissipative effects.

† The authors have discovered two editorial errors in the paper by Kuznetsov *et al.* (1978) which need to be brought to the attention of others who wish to use that important work. In their table 1, the column labelled ΔP_0 (the initial amplitude of the pressure perturbation) actually contains the values of ΔP_m (the measured amplitude of the pressure perturbation), as can be verified from their figure 10. In addition, their equation (7), which reads

$$u_0(\xi) = \Delta P_0(\xi)/\rho_0 c_0,$$

should actually read $$u_0(\xi) = [1+(\gamma+1)/2\phi_0]\Delta P_0(\xi)/\rho_0 c_0,$$

which can be verified by going through the derivation of their equation (9). When the corrected equation for $u_0(\xi)$ is used, the values of the parameter σ that they give can be verified approximately using the data which they quote. However, the authors have been unable to verify the values of the Reynolds number Re that they quote. The latter discrepancy remains unresolved.

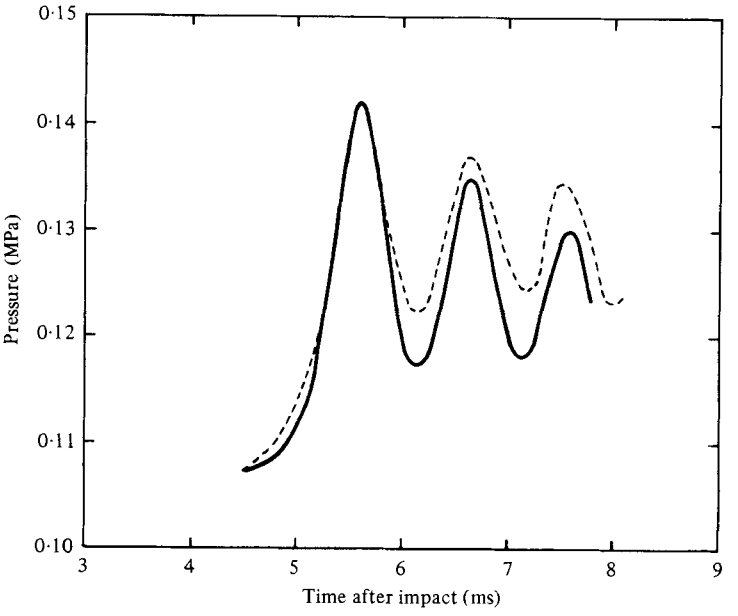

FIGURE 3. Comparison of theory to case B in Kuznetsov *et al.* (1978); corrected $\Delta P_0 = 0{\cdot}35/1{\cdot}28 = 0{\cdot}27$. ——, theory; – – –, data.

The bubbles had a nominal diameter of 1 mm and a nominal volume concentration of 1 percent.

For the most part, the material properties that were necessary for the computations were listed by Kuznetsov *et al.* (1978). Exceptions included the bulk modulus of the liquid and the specific heats of the gases. The values that have been assumed are $K = 2{\cdot}66$ GPa, $C_{pg} = 870{\cdot}6$ J/kg K (for carbon dioxide), and $C_{pg} = 5133{\cdot}3$ J/kg K (for helium). The thermal conductivities of the gases could be computed from the specific heats and the listed values of the thermal diffusivities.

Calculations were made for all the carbon dioxide experiments and for the short-propagation-distance helium experiments. The results are compared to the experimental data in figures 3–10.† These figures are plots of pressure versus time after impact. Since absolute time scales were not included with the data, the data were time-shifted until the peak pressures coincided. The theoretical results contain only plots of the gas pressure. The liquid pressure is not plotted. The computed liquid-pressure profiles consisted of very rapid oscillations about a value equal to the gas pressure. The pressure transducers that were used in the experiments would not have resolved these oscillations. Consequently the data has been compared to the computed profile of the gas pressure.

Figures 3–5 compare the calculations to the data for the propagation distance of 0·6 m with carbon dioxide bubbles. These figures correspond to cases B, E, and F in Kuznetsov *et al.* (1978) which produced three different wave-propagation profiles; a

† The large times required by these calculations necessitated the use of a continuous rezoning technique whereby the grid spacing was constantly altered so as to concentrate mesh points near computationally active regions (Lawrence & Mason 1971). Thus, the portions of the tube which were at uniform pressures contained large mesh spacings and the portions of the tube with large pressure gradients contained mesh spacings as small as 0·5 mm.

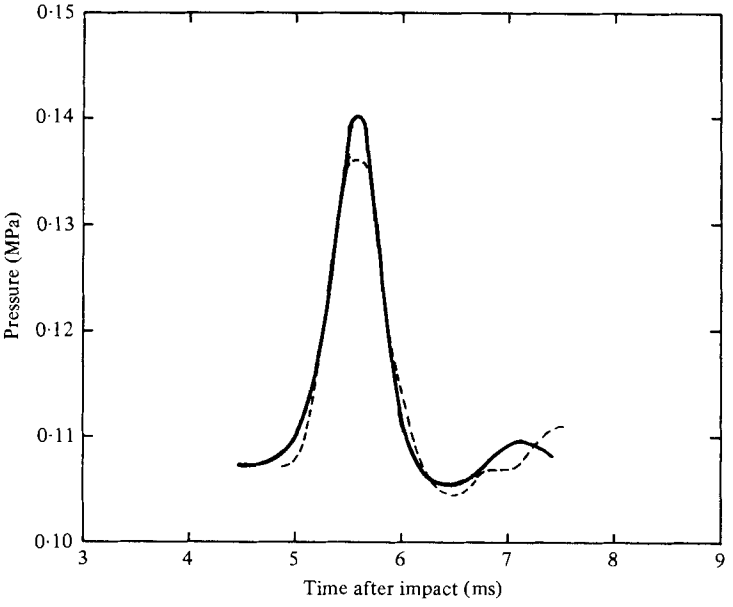

FIGURE 4. Comparison of theory to case E in Kuznetsov *et al.* (1978); corrected $\Delta P_0 = 0{\cdot}29/0{\cdot}61 = 0{\cdot}48$. ——, theory; – – –, data.

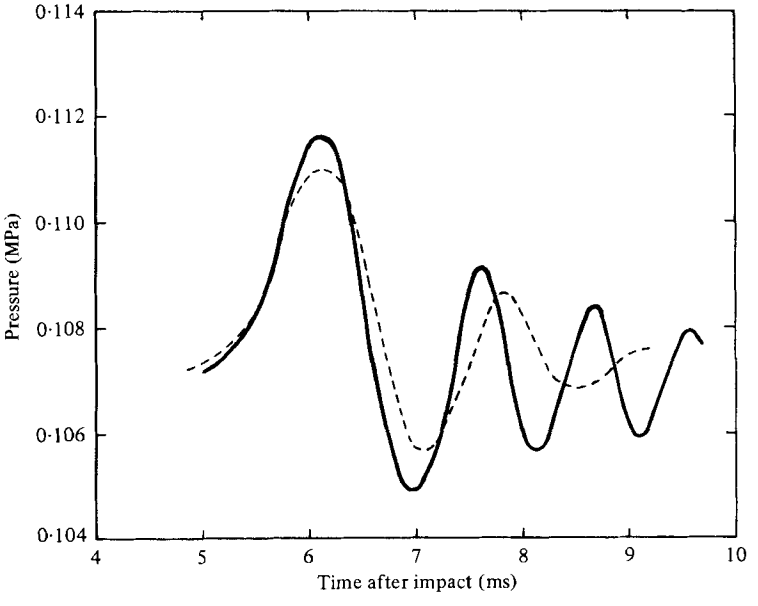

FIGURE 5. Comparison of theory to case F in Kuznetsov *et al.* (1978); corrected $\Delta P_0 = 0{\cdot}035/0{\cdot}28 = 0{\cdot}13$. ——, theory; – – –, data.

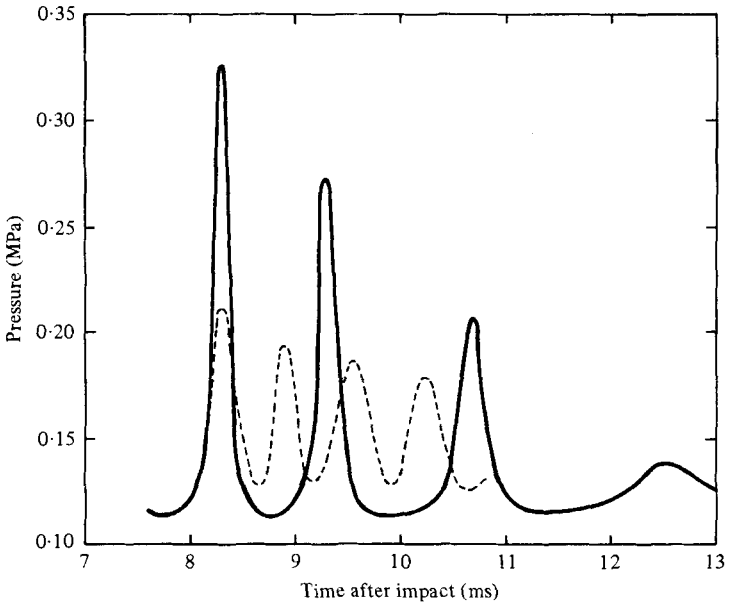

FIGURE 6. Comparison of theory to case *C* in Kuznetsov *et al.* (1978); corrected $\Delta P_0 = 0{\cdot}945/0{\cdot}34 = 2{\cdot}78$. ——, theory; – – –, data.

shock wave, a single soliton, and a wave packet.† The agreement between theory and experiment is seen to be quite good in all three cases.

Case *E*, figure 4, was also compared to a calculation in which thermal damping was modelled by the acoustic approximation discussed in § 4 (van Wijngaarden 1972*a*). In that calculation almost no attenuation of the peak amplitude was observed.

Figures 6 and 7 compare theory and experiment for the longer propagation distance of 1·4 m with carbon dioxide bubbles. Here the comparison shows greater error. Figure 6 corresponds to case *C* in Kuznetsov *et al.* (1978), where the wave propagation profile was characterized as a multiple soliton. The theoretical results do exhibit the solitary-wave structure suggested by this characterization; however, the experimental data are more objectively characterized as a wave-packet form, in that the oscillations are not as distinctly separated as in the calculations.

Kuznetsov *et al.* (1978) stated that in some of their experiments atomization or breakup of the bubbles occurred. Since the pressure perturbation in case *C* was much larger than in the previous experiments, atomization of the bubbles may have occurred in this case. The calculations of course would not account for this effect.

The same discrepancy is seen in figure 7 (case *D*). Again, higher pressures were used; however, in this case the theory does exhibit the two-soliton description assigned to this experiment.

In figures 8–10 the results are compared to the data for helium bubbles, corresponding to cases 13*a*, 14*a* and 15*a*. These results were obtained at the 0·6 m data station. The qualitative agreement between the calculations and the data is good; however, the theory underpredicts the period of the oscillation behind the wave front. High

† These classifications are based on the perturbation results presented by Kuznetsov *et al.* (1978).

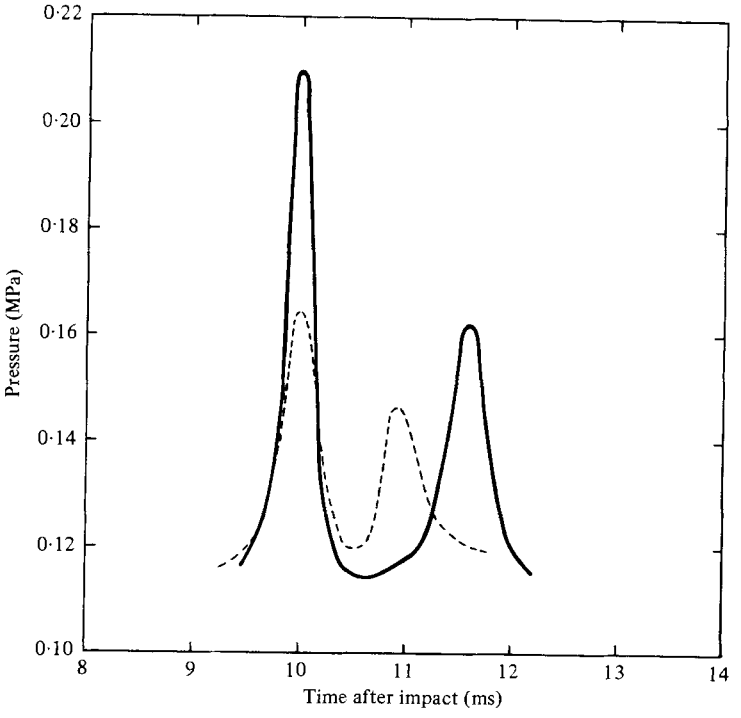

FIGURE 7. Comparison of theory to case *D* in Kuznetsov *et al.* (1978); corrected $\Delta P_0 = 0{\cdot}48/0{\cdot}37 = 1{\cdot}3$. ——, theory; – – –, data.

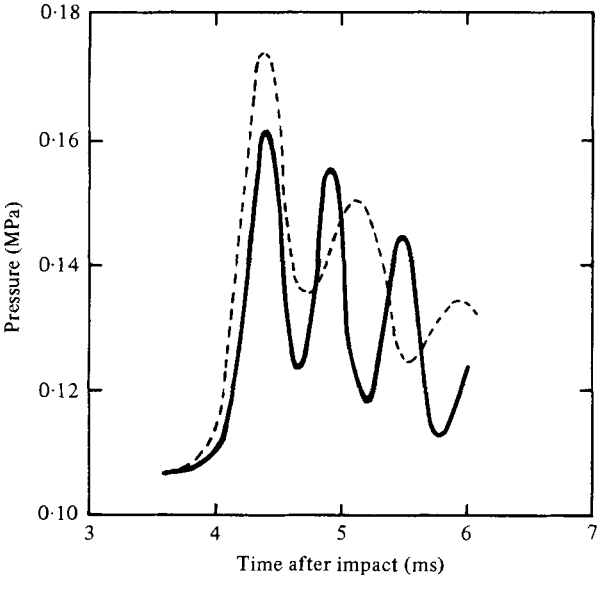

FIGURE 8. Comparison of theory to case 13*a* in Kuznetsov *et al.* (1978). ——, theory; – – –, data.

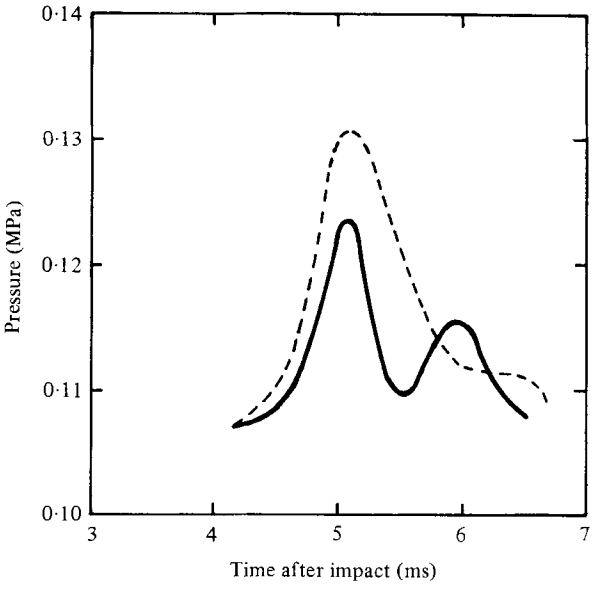

FIGURE 9. Comparison of theory to case 14*a* in Kuznetsov *et al.* (1978). ——, theory; – – –, data.

pressures were used in these experiments; however, the data in these cases does not seem to exhibit any evidence of possible bubble breakup.

Figures 11 and 12 are comparisons with the data of Noordzij & van Wijngaarden (1974). In these experiments a pressure pulse was generated by first covering the top of the tube with a diaphragm and partially evacuating the interior of the tube. The diaphragm was then broken. Consequently the pressure-pulse amplitudes used in these experiments are much lower than in the previous experiments. In these experiments the bubbles consisted of air.

Again, only the computed gas-pressure profiles are plotted. Figure 11 shows the comparison at a propagation distance of 0·2 m. Both the quantitative and qualitative agreement are seen to be poor. Two theoretical calculations are shown. The first includes the dissipation mechanisms described in this work. The second excludes all damping terms. Even in the complete absence of damping the calculations do not exhibit the oscillations that appear in the data.

For the air–water system and the low pressures used in these experiments, the perturbation analysis presented by Kuznetsov *et al.* (1978) suggests that the results of this experiment should correspond to the theoretical profile that was obtained. It is worth noting that these data were collected at a point that was slightly less than four tube diameters from the surface where the pressure pulse was applied. Any lack of planarity in the wave could result in an oscillation with a period equal to two transit times across the tube diameter. The tube diameter in this case was 55 mm, and the wave-propagation speed was 68 m/s, which corresponds to a period of 1·6 ms. This is very close to the experimentally observed oscillation period.†

† The authors recommend that future experiments be designed so that the resonance period of the bubbles differs significantly from the time required for the wave to propagate across the diameter of the tube.

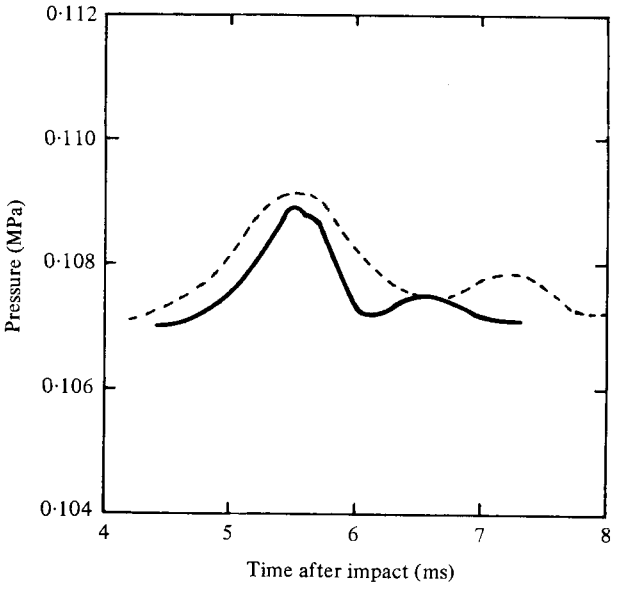

FIGURE 10. Comparison of theory to case 15*a* in Kuznetsov *et al.* (1978). ——, theory; – – –, data.

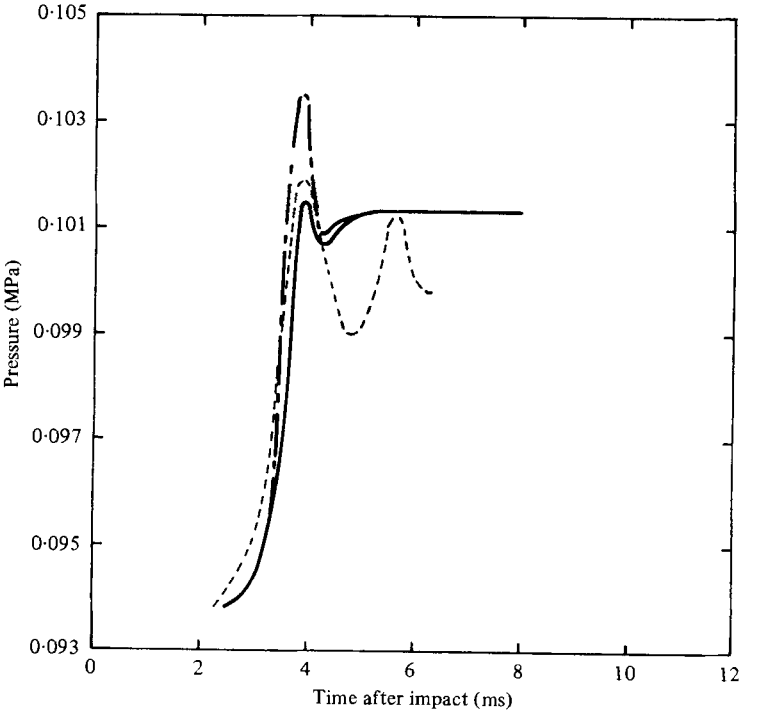

FIGURE 11. Comparison of theory to region *A* data, figure 2, Noordzij & van Wijngaarden (1974). ——, theory; — · —, theory without damping; – – –, data.

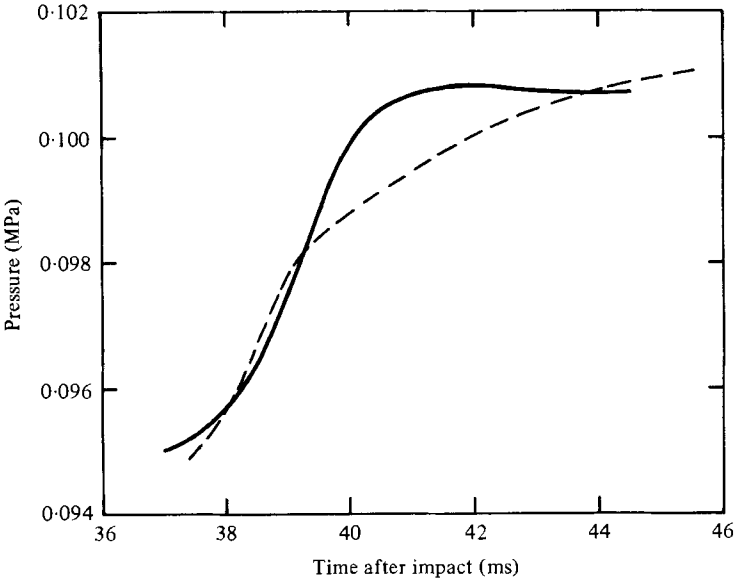

FIGURE 12. Comparison of theory to region *B* data, figure 2, Noordzij & van Wijngaarden (1974). ——, theory; – – –, data.

Noordzij & van Wijngaarden (1974) defend the appearance of these oscillations in their data by citing the work of Crespo (1969). Crespo's theoretical calculations, which were based on incompressible-liquid behaviour, do exhibit oscillations on a time scale which is compatible with their experimental data. However, his calculations were for much larger amplitude waves, with pressure ratios 50 times greater than the experiment. His analysis for the case of weak pressure waves did not predict oscillations behind the wave front. There is apparently no direct conflict between the present calculations and those of Crespo in that neither calculation predicts large oscillations at low pressure amplitudes. Consequently, the suggestion that the experimentally observed oscillations are predicted by calculation appears to the authors to be open to question.

Figure 12 presents the comparison for the 2·5 m data station. Here no oscillations are seen in either the data or the calculations; however the data exhibits an additional damping effect not occurring in the calculations. Noordzij & van Wijngaarden attribute this additional damping to the effect of relative motion between the liquid and the bubbles, which was not modelled in the present investigation.

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